Trig-substitution.

Q1: Substitute the x-terms in $\int \frac{x^2}{\sqrt{4-x^2}}dx$ by $\boxed{x = 2\sin\theta}$ and express the integral in terms of θ.

Q2: Use double angle formula

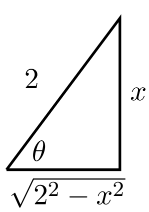
$$\boxed{\sin^2 \spadesuit = \frac{1-\cos 2\spadesuit}{2}}$$

to evaluate the integral $\int 4\sin^2\theta d\theta$

Q3: If $\boxed{x = 2\sin\theta}$, find (express) $\boxed{2\theta - \sin(2\theta)}$ in terms of x by using the formula

$$\boxed{\sin(2\clubsuit) = 2\sin\clubsuit\cos\clubsuit}$$

and the following right triangle:



Q4: Evaluate $\int \frac{x^2}{\sqrt{4-x^2}}dx$

Answer to Q1: If $\boxed{x = 2\sin\theta}$, then

$$\sqrt{4-x^2} = \sqrt{4-(2\sin\theta)^2} = \sqrt{4(1-\sin^2\theta)} = \sqrt{4\ \cos^2\theta} = 2\cos\theta \tag{0.1}$$

$$dx = d(2\sin\theta) = 2\cos\theta\ d\theta \tag{0.2}$$

Plug $\boxed{x = 2\sin\theta}$, $\boxed{(0.1)}$ and $\boxed{(0.2)}$ into $\int \frac{x^2}{\sqrt{4-x^2}}dx$, we have

$$\int \frac{x^2}{\sqrt{4-x^2}}dx = \int \frac{(2\sin\theta)^2}{\boxed{\sqrt{4-(2\sin\theta)^2}}} \cdot \boxed{2\cos\theta\ d\theta} = \int \frac{(2\sin\theta)}{\boxed{2\cos\theta}} \cdot \boxed{2\cos\theta\ d\theta} = \boxed{\int 4\sin^2\theta d\theta}$$

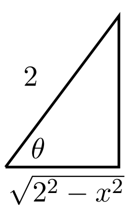
Answer to Q2: According to **DOUBLE ANGLE FORMULA** $\sin^2 ♠ = \frac{1-\cos 2♠}{2}$, we have

$$\int 4 \cdot \boxed{\sin^2\theta} d\theta = \int 4 \cdot \boxed{\frac{1-\cos 2\theta}{2}} d\theta = \int 2 - 2\cos 2\theta\ d\theta \quad = \quad \int 2\ d\theta - \boxed{\int 2\cos 2\theta\ d\theta}$$

$$= \quad 2\theta - \boxed{2 \cdot \frac{1}{2}\sin 2\theta} + C$$

Answer to Q3: Recall in a right triangle, $\sin\theta$ is defined to be ration of the length of the side that is **opposite** angle θ to the length of the **hypotenuse**, i.e.,

$$\sin\theta = \frac{\textbf{opposite}}{\textbf{hypotenuse}}$$



If $\boxed{x = 2\sin\theta}$, then $\sin\theta = \frac{x}{2}$. Then we can draw such a right triangle, with **opposite** $= x$, **hypotenuse** $= 2$, and the **adjacent** $= \sqrt{2^2 - x^2}$ (can be computed by Pythagorean theorem.)

Therefore,

$$\cos\theta = \frac{\textbf{adjacent}}{\textbf{hypotenuse}} = \frac{\sqrt{2^2-x^2}}{2}$$

Also, $\sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}(\frac{x}{2})$. Finally, combined with **DOUBLE ANGLE FORMULA** $\sin(2\theta) = 2\sin\theta\cos\theta$, we have

$$2\theta - \sin(2\theta) = 2\theta - 2\cdot\sin\theta\cdot\cos\theta = 2\sin^{-1}(\frac{x}{2}) - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{2^2-x^2}}{2}$$

Answer to Q4: Combination of Q1-Q3: If $x = 2\sin\theta$, then

$$\int \frac{x^2}{\sqrt{4-x^2}}dx \overset{Q1}{=\!=} \int 4\sin^2\theta d\theta \overset{Q2}{=\!=} 2\theta - \sin 2\theta + C \overset{Q3}{=\!=} 2\sin^{-1}(\frac{x}{2}) - 2\cdot\frac{x}{2}\cdot\frac{\sqrt{2^2-x^2}}{2} + C$$

Q: Evaluate

$$\int \frac{dx}{(\sqrt{4+x^2})^3}$$

Step1: Simply the integral

$$\int \frac{dx}{(\sqrt{4+x^2})^3}$$

by certain trig-substitution (by setting x to be some trigonometric function and get rid of $\sqrt{4+x^2}$).

Step2: Evaluate the integral $\int \frac{1}{4}\frac{1}{\sec\theta}d\theta$

Step3: If $x = 2\tan\theta$, find (express) $\frac{1}{4}\sin\theta$ in terms of x.

Details of Step1: Recall **Trig-Identity** $1+\tan^2\theta = \sec^2\theta$. If $\boxed{x = 2\tan\theta}$, then

$$\sqrt{4+x^2} = \sqrt{4+(2\tan\theta)^2} = \sqrt{4(1+\tan^2\theta)} = \sqrt{4\ \sec^2\theta} = 2\sec\theta \tag{0.3}$$

$$dx = d(2\tan\theta) = 2\sec^2\theta\ d\theta \tag{0.4}$$

Plug (0.3) and (0.4) into $\int \frac{dx}{(\sqrt{4+x^2})^3}dx$, we have

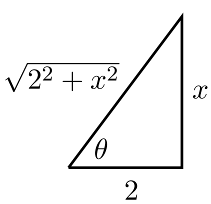
$$\int \frac{dx}{(\sqrt{4+x^2})^3} = \int \frac{2\sec^2\theta\ d\theta}{(2\sec\theta)^3} = \int \frac{1}{4}\frac{1}{\sec\theta}d\theta$$

Details of Step2: According to **Trig-Identity** $\sec\theta = \frac{1}{\cos\theta}$, we have

$$\int \frac{1}{4}\frac{1}{\sec\theta}d\theta = \int \frac{1}{4}\cos\theta d\theta = \frac{1}{4}\sin\theta \quad + C$$

Details of Step3: Recall

$$\tan\theta = \frac{\textbf{opposite}}{\textbf{adjacent}}$$



If $x = 2\tan\theta$, then $\tan\theta = \frac{x}{2}$. Then in the right triangle:

opposite $= x$, **adjacent** $= 2$, and

the **hypotenuse** $= \sqrt{2^2+x^2}$ (can be computed by Pythagorean theorem.)

Therefore,

$$\sin\theta = \frac{\textbf{opposite}}{\textbf{hypotenuse}} = \frac{x}{\sqrt{2^2+x^2}}$$

Then, $\frac{1}{4}\sin\theta = \frac{1}{4}\frac{x}{\sqrt{2^2+x^2}}$.

Answer to Q: If $x = 2\tan\theta$, then

$$\int \frac{dx}{(\sqrt{4+x^2})^3} \overset{Step1}{=\!=} \int \frac{1}{4}\frac{1}{\sec\theta}d\theta \overset{Step2}{=\!=} \frac{1}{4}\sin\theta \quad + C \overset{Step3}{=\!=} \frac{1}{4}\frac{x}{\sqrt{2^2+x^2}} + C$$